Chapter 12
Relativity and Zeno's Paradoxes

The word “paradox” is sometimes found in discussions of relativity. Not having thought the problem through, we may often find one of the predictions of relativity to be paradoxical. One such paradox is that of the long car and the short barn. A barn has doors on opposite sides and a car is approaching the barn at a fast enough speed that it contracts appreciably. The car at rest would be longer than the distance between the doors, but contracted in this way, someone in the barn could close the doors momentarily with the car inside and then open them again before the car smashes into the fore door. The apparent paradox is that the car should find the barn contracted as well and should not be able to fit inside. The resolution of this apparent paradox lies in the relativity of simultaneity. In other words, from the point of view of the car, the doors do not close at the same time and one door is always open. First, the fore door closes and then opens again; then as the front end of the car passes through the fore door, the rear door closes.

The Greek philosopher Zeno of Elea lived in the 5^{th} century BC. He is credited with a number of paradoxes concerning the problem of motion. These paradoxes were restated and refuted by Aristotle (384-322 BC) in *Book VI* of his *Physics*.

The Arrow Paradox suggests that any kind of motion is illusory, if not impossible. As recorded by Aristotle, Zeno's argument is,

> “If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless.”

In other words, if an object such as an arrow occupies a particular space equal to its own volume, without moving outside that space, it is at rest. At any “moment” (an infinitesimal interval of time), Zeno argues, the arrow occupies a space exactly equal to its volume and does not move beyond it. The conclusion is that the arrow is at rest in that moment, and all preceding and following moments; it is motionless. This argument would probably have used high-speed photography as an example if the technology had existed at the time. Even a bullet in flight appears motionless when caught on high-speed film. If time is a series of frozen instants, then motion is an illusion, like a movie film played on a screen at 24 frames per second.

There are many possible arguments against Zeno's reasoning. It can be argued that Zeno's fallacy is that one cannot determine from one instant, from a single photograph, whether an object is in motion; that motion can only be determined by comparing on instant to the other. But rather than being Zeno's fallacy, this is precisely his point. One is tempted to argue that between any two moments there must be another moment, and thus time is continuous rather than being “quantized,” or broken up into movie frames. But we'll leave that argument aside for now and instead focus on Zeno's supposition of an infinitesimal interval of time, which would be the inverse of an infinite interval. The mathematical utility of “infinity” is the fact that it does not obey the rules of addition, multiplication, and so on when used with other numbers.

In this sense it is not a number so much as a representation of impossibility or a warning sign. We could reject Zeno's argument on these grounds, supposing that if he had chosen any *finite* fraction of time in which to examine the arrow's motion, he would find a proportionally small – but non-zero – distance traveled by the arrow. In this case, we would essentially be arguing that *space* is continuous rather than "quantized," or broken up into discrete, indivisible distances. Let's also leave that argument aside for now and instead appeal to special relativity. We need not assume that either space or time is infinitely divisible. Einstein might have resolved the problem thus:

> "If anything occupies a space equal to the space it occupies at rest, it is at rest. That which is in locomotion occupies a lesser space in a manner dependent on the velocity of its motion, and thus this velocity can be determined at any instant from the object's proportion."

When the arrow is in flight, it is measured by objects at rest (or in relative motion to the arrow) to be shorter from end to end. If an arrow one meter long could be shot at half the speed of light and were captured on extremely high-speed film next to a stationary measuring stick, the arrow would measure only 87 centimeters. By measuring the length of the arrow, we can calculate its speed relative to the measuring stick. Thus the paradox is resolved.

Another of Zeno's paradoxes concerns a hypothetical footrace between the Greek hero Achilles and a tortoise. Confident in his superiority, Achilles gives the tortoise a head start. With a head start of 100 meters, Achilles might reach the tortoise's starting point quickly, but in that time the tortoise will have moved on, however slowly and slightly. At that point it is as if another race had begun, only with a shorter head start for the tortoise. Though the head start may grow smaller with each repetition, Zeno argued that in this situation Achilles could never catch up to the tortoise, no matter how fast he may run, because there would be an infinite number of distances to catch up. Aristotle recorded Zeno's race paradox thus:

> "In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."

This may seem to be the sillier of the two paradoxes, but it is at least as interesting as the other. Imagine that Zeno, having long ago departed to some other state of existence, has endeavored to catch up with current developments in science. Not yet having succeeded, he meets Einstein.

Einstein: Hello, Zeno.
Zeno: Hello. How do you know my name?
Einstein: I had reason to expect you. What are you reading?
Zeno: A book which states that laws of physics are observed in any frame of reference; that the velocity of an observer is only meaningful in comparison to other frame.
Einstein: Oh, I wrote such a book.
Zeno: Is it true that it is impossible that any object will be measured to have a speed greater than the speed of light?

Einstein: It is so.
Zeno: Suppose an observer K is able to throw a stone at half the speed of light. If K is in motion relative to another observer K' at three-quarters of the speed of light and attempts to throw a stone at half the speed of light, does he find himself prevented due to his relative motion? Does this not violate the principle of relativity, which states that one cannot determine strictly within one's own frame his velocity with regard to another frame?
Einstein: No, K will find that his stone has a speed of .5c, because he measures himself to be at rest. However K' will measure the stone to have a speed of . . . (makes quick use of a pocket calculator) .91c.
Zeno: I read earlier that two frames of reference will measure the velocity of an object along the axis of their relative motion to vary arithmetically with their relative motion. In other words, if K measures the stone to have velocity u of .5 and has a relative velocity v of .75 with respect to K', then K' will measure the stone to have a velocity u' of $u+v$, which is 1.25. How is it that K and K' disagree on the speed of the stone even when using Galilean transformations to compare measurements?
Einstein: It is because they must agree on the speed of light, c, for the laws of physics to apply equally to the both of them. From this foundation it must eventually follow that K' will measure a velocity u' of $\dfrac{u+v}{1+\dfrac{uv}{c^2}}$.
Zeno: How very queer. Suppose K is in a rocket ship having enormous amounts of fuel. Is it also true that K cannot himself accelerate to a speed equal to or greater than c?
Einstein: Yes, for reasons of conservation of mass, which is covered in a later chapter. As K approaches the speed of light, he is measured to be more massive. At speeds near c, this mass becomes so large that it would require near-impossible amounts of energy to accelerate it.
Zeno: But who tells K that he is near c and therefore more massive? Does this not violate relativity? Does he not measure himself to be at rest and therefore able to accelerate freely?
Einstein: Special relativity deals with *inertial* frames, frames of reference in uniform motion, not acceleration. General relativity, which concerns accelerating reference frames, is discussed in a later chapter, but this should not hinder us in this case. Indeed, K will notice no difference if he measures strictly within his own frame of reference. But if he compares his frame of reference to that of K', he will find that the difference between the two can never exceed c. From the point of view of K', K has become too massive to be accelerated by an external force; furthermore, K's internal systems, his clocks, have slowed to the point where the rate of fuel supplied internally for acceleration is near zero. From the point of view of K, he enters a new inertial frame of reference with each burst of acceleration provided by his rocket engines. And from moment to moment he is able to accelerate at the same rate within each successive frame of reference, regardless of his velocity with respect to K'. But he finds that as he approaches a relative velocity c with K', his rate of acceleration with respect to K' becomes slower. He is no longer accelerating efficiently with respect to K', but only with respect to each successive reference frame K.

Zeno: And if K' were to send a beam of light past K, K' would never be able to catch up to it?
Einstein: K can never catch up to the light beam.
Zeno: Are you familiar with my work?
Einstein: Why, yes, I just happened to read a reference to it on my way here.

The problem just discussed between Zeno and Einstein is "relativistic addition of velocities," the problem of adding one velocity to another when one of those velocities is a significant fraction of the speed of light. Such velocities cannot simply be added arithmetically as allowed by Newtonian physics. A given velocity *u* when added to a velocity of zero will result in the velocity *u.* But u cannot be added to velocity v to get the value u+v. Let us imagine the frames J and K, in relative motion at .75c. Anne, an astronaut in J, and Bette, an astronaut in K, are in uniform motion in space. Bette fires her rocket engines for a period of time, turns them off again, and then measures that his velocity has increased from 0 to .5c. She has accelerated out of her former inertial reference frame K and is now in a completely different inertial frame of reference, L. She measures a difference of *v* between his current and former reference frames, K and L. But she does not measure the same difference v in the increase of relative velocity with Anne in J. She sees the relative velocity of J increase from .75c to .91c, a difference of only .16c.

Figure 12-1. $$\frac{u+v}{1+\frac{uv}{c^2}}=\frac{.75+0.5}{1+\frac{.75*0.5}{1^2}}=\frac{1.25}{1.375}=0.91$$

I would liken the problem of relativistic addition of velocities to the game of tetherball. As the ball swings around the pole on its rope, your goal is to speed it on its way, so you swat at it with your arm each time it comes around. But if your arm swings to reach the ball's path before or after the ball is there, you will miss. The swing of your arm must be in phase with the revolution of the ball. Not only must your timing be correct, but the angle of your swing must match the angle of the ball's swing in order for you to strike it efficiently. If your arm is swinging west and strikes a north-going ball, it is not nearly as efficient as if it strikes the ball when it is also headed west. The direction and timing of your swings must be *in phase* with the swing of the ball.

In the example of relativistic addition of velocities above, we might say that J , K, and L are *out of phase*. When Bette's rockets begin to fire, she is in phase with reference frame K, and is able to efficiently "push off" from that frame of reference into another one. But she is not in phase with J. When she accelerates, she is not accelerating so much with respect to J as she is to K.

The most famous "paradox" associated with relativity is the so-called "Twin Paradox." This concerns two theoretical twins, one of whom leaves Earth for a journey in space at speeds near the speed of light. The astronaut twin returns to Earth to find that more time has passed on Earth than in the spacecraft and that the stay-at-home twin is several years older.

Suppose that Achilles and the Tortoise have a race to collect a piece of the rings of Jupiter and return to Earth. Achilles is able to reach 99.9 percent of the speed of light, or .999*c*, while the Tortoise is able only to travel 50 percent of the speed of light, .5*c*. The race begins, and as Achilles leaves Earth orbit, he finds that space contracts in the direction of his motion and that Jupiter is closer than he thought. Because it takes him a little less than half a minute to accelerate to top speed or to slow down again, he is able to complete the journey in only a minute. He realizes that once he reaches his top speed, he can cover any distance at all in mere seconds because of the relativistic contraction of space. Knowing that the Tortoise enjoys no such benefit at a mere .5*c*, he decides to press his advantage and collect a sample not only from the rings of Jupiter, but from those of Saturn and Neptune as well. Having done so, he looks at his watch and sees that less than five minutes have passed since beginning the race. He returns to Earth with his trophies, expecting to revel in a spectacular victory, but finds that the Tortoise has returned several hours ahead of him even though the Tortoise by his own reckoning was gone for hours.

How is it that Achilles has lost this race? Having gone twice as fast as the tortoise, he has nevertheless gone several times the distance and must naturally lose. Though space has contracted for him, his clock has slowed down proportionally. An observer from Earth would measure that the Tortoise's journey lasted between 2 and 3 hours, while Achilles' took the better part of a day.

That's fine, you may say, but from each contestant's point of view, haven't they each had zero velocity with respect to their own frame of reference? In the case of the twins, isn't it just as valid for the space-going twin to suppose that she is stationary and the earthbound twin accelerates away and comes back? In that case, shouldn't they age at the same rate? The answer is that the twins' situations are not equal, and neither are those of Achilles and the Tortoise. Achilles and the space-going twin each experienced an acceleration much greater in magnitude and/or duration than their counterpart. Let's call the astronaut twin Anne and call her twin Bette. Anne was pressed back into the seat of her rocket as she blasted off into space and again when her rocket turned around to return to earth. Bette felt no such acceleration. This is what makes their frames of reference non-equivalent and allows us to designate which of the two will age more quickly. But we are getting somewhat ahead of ourselves; Special Relativity deals with inertial frames of reference, not accelerating ones. We'll explore acceleration more in depth with General Relativity, which will not be covered in this volume. For now, suffice it to say that these differences in aging rates are measurable with atomic clocks and have been verified experimentally by comparing a clock which has been in high-speed flight to one which has been kept on the ground.