Chapter 3
Perspective in Mathematics

In our early years, if we are inquisitive enough, we may wonder why five plus three has the same answer as three plus five. At one point in our education, this symmetry is pointed out as the *commutative law* of addition, a term which most of us soon forget because we have no use for it. I like math best when I can visualize it. One teaching method for arithmetic is the use of the *number line*, something which your elementary-school math teacher may have drawn on the blackboard.

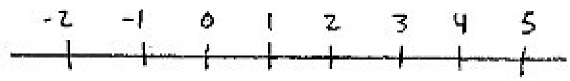

Figure 3-1. The number line.

A number line is simply a line with regular distances marked off, each mark having a whole number (*integer*) which is one greater than the one to its left. Somewhere in the middle of the number line is zero, with the positive and negative integers on either side of it which increase radially in absolute value. On this number line, we can learn the basics of addition and subtraction. Each number to be added or subtracted is simply a number of steps to the left or right. Two steps to the left of six we find four, and so on. This relates our concept of addition and subtraction to our ideas of position and length. Let's return to the idea of the commutative law using our number line. Supposing that we always count from left to right, we place three green apples on the line and then five red ones to the right of them to represent three plus five.

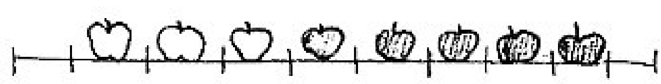

Figure 3-2.

We then can see that if we abandon convention and count from right to left, we have a perfect representation of five plus three. This is not the point I wish to make. Imagine that our blackboard is instead a transparent pane of glass and that someone is on the opposite side. We'll call her Jane. If Jane counts from left to right, she will be counting five plus three, whereas in the same left-to-right direction from our side, we count three plus five. The point I wish to make is that which of the two problems we are looking at – five plus three or three plus five – is a matter of perspective. With a little bit of further contemplation, we can spatially visualize the interrelation of several different mathematical facts:

3+5=8
5+3=8
8-3=5
8-5=3

This reversal of perspective, or the fact that three plus five on our side of the glass looks like five plus three to Jane, has even more mathematical significance. A reversal of *direction* means a reversal of *value*. Let me explain what I mean by this. We count movement to the right as positive and movement to the left as negative. If we label the leftmost apple as “start” and the rightmost apple as “finish,” the difference between the two is eight. We add eight apples to get from the start to the finish. To Jane, the difference is a negative eight. She subtracts eight apples to get from start to finish. The difference in perspective is a difference both in direction (left or right) and in relative value (positive or negative).

As another example, let's imagine that we pass six dollars to Jane through some sort of window. She posts them on the number line right next to the zero mark, the location of which is agreed on by both of us. She places them to the right of the zero, and thus counts the number of dollars she has gained. The number line is marked on her side of the glass with increasing values from left to right, so the last dollar reaches the six mark. But from our point of view, the dollars are on the *left* of the zero mark. The number line on our side is also marked with increasing values from left to right, so the last dollar reaches the minus six mark, indicating that we have “gained” minus six dollars, or in other words, that we have six fewer dollars.

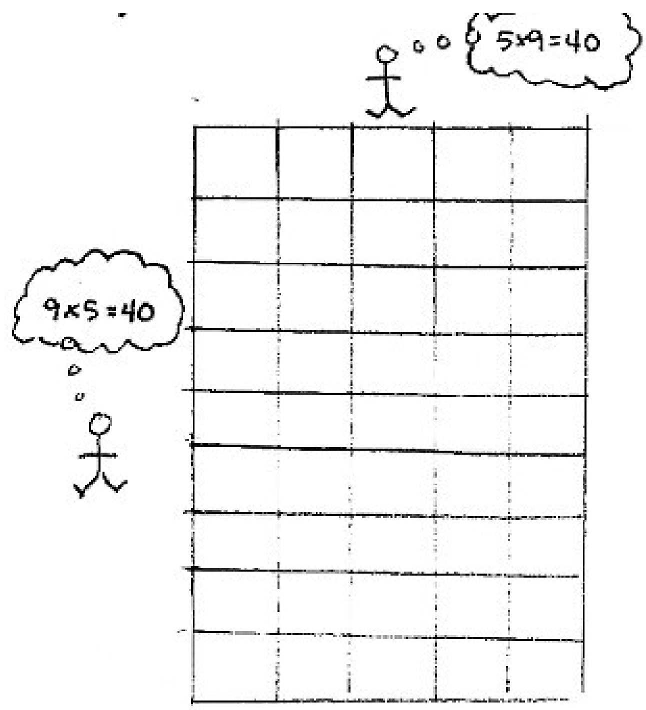

Figure 3-3.

In our basic mathematical education, after mastering addition and subtraction, we are introduced to the operations of multiplication and division: “If Sam has five baskets and each basket contains nine apples, how many apples does Sam have?” I remember being mystified to find that the answer was the same if there were *nine* baskets with *five* apples each. A line is a *one-dimensional* construct, having no depth or width to go with its length. If we line up the baskets on our number line, it does not help us solve this new mystery. Multiplication and division are *two-dimensional* operations. We need to arrange the apples in rows and columns to see the symmetry in the commutative property of multiplication. Whether the baskets enclose rows or columns of apples, the arrangement of the apples is the same. Perhaps it works better for you to take the apples out of the baskets altogether and arrange them in ranks, five by nine. Depending on which side of the arrangement you stand on, there are either five rows or five columns. It depends on your perspective (Figure 3-3).

Again, there are several interrelated mathematical facts represented in Figure 3-3:

$5 \times 9 = 45$
$9 \times 5 = 45$
$45 \div 9 = 5$
$45 \div 5 = 9$

For thousands of years, thinkers of many kinds have had a special affection for the circle: artists, philosophers, and mathematicians alike. It represents perfection in its roundness and eternity in its continuity. Its cousin the *sine wave* is also one of the most beautiful and profound forms in all of mathematics. Here is an example:

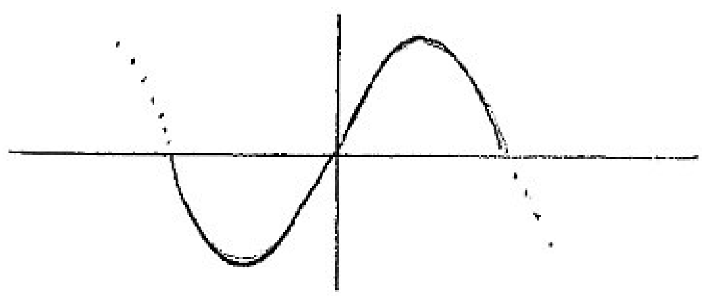

Figure 3-4. A sine wave.

As we will see in the following examples, this wave represents a continuous, progressive, and circular relationship between something and its opposite. It represents a rotation in perspective between forward and backward, clockwise and counterclockwise. It is also the expression of many natural phenomena. For example, sinusoidal vibrations of the air create pure tones in the ear. The yearly progression of the sun back and forth across the equator can be charted as a sine wave (see Appendix A).

Circular motion can be defined by two complementary sine waves. Let's imagine a piece of chewing gum stuck to the end of the second hand on our classroom clock, circling the center of the clock but remaining a constant six inches away. The clock is hanging above the doorway. If we sit anywhere in the classroom, the gum's motion appears roughly circular. If we stand in the doorway and look up, however, the gum appears to swing back and forth from the left and right sides of the clock. Let's establish a coordinate system to define the gum's motion. If you're not familiar with the idea of coordinates, you may want to come back after having read chapter four, but you may be able to follow along just fine. Here we'll use a two-dimensional rectangular coordinate system with the origin being the center of the clock, the x axis running horizontally through the three and nine o'clock positions, and the y axis running vertically through the six and twelve o'clock positions. We can draw a circle around the origin which is the gum's path around the clock, but this does not tell us where the gum is at any given time (Figure 3-5).

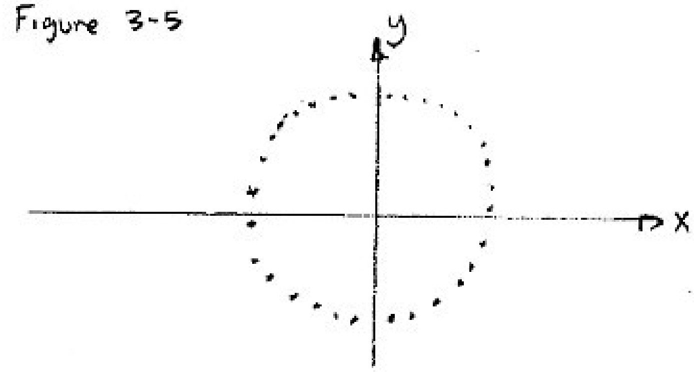

Figure 3-5. A two-dimensional graph of the gum's movement.

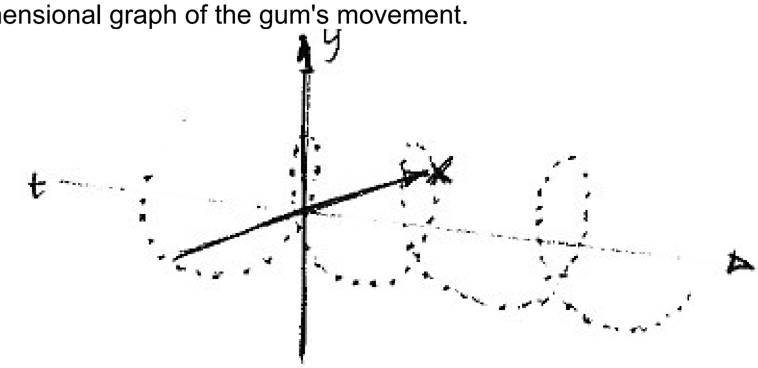

Figure 3-6. A three-dimensional graph of the gum's movement.

We could add a third axis for time, pointing perpendicularly through the clock's center, and then we could track its path over time. The path of the gum would be a helical graph, a corkscrew shape curling around the time axis (Figure 3-6). If we want to restrict our charting to two dimensions, though, we can do it with two sine wave graphs. One (Figure 3-7) will chart the gum's movement from left to right, and another (Figure 3-8) will track its movement up and down.

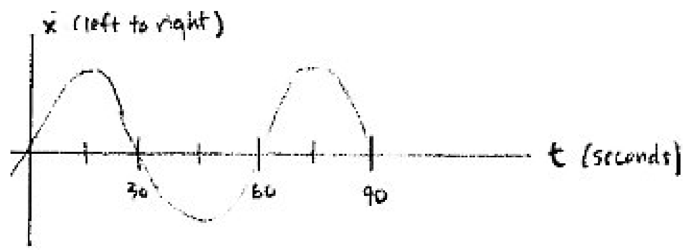

Figure 3-7. *x=sin(6t)*

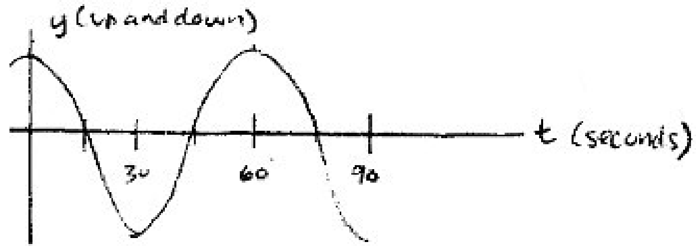

Figure 3-8. *y=cos(6t)*

The graph of the gum's movement left and right is called a sine function and the graph of the up and down movement is the cosine function. Each graph is a function of time, with a full cycle being completed every minute. At first glance these two graphs may look the same, but it is important to note that they don't line up in time. We call these two functions of time *sin(t)* and *cos(t).* They have a relationship with each other as well as with the circular motion they describe.

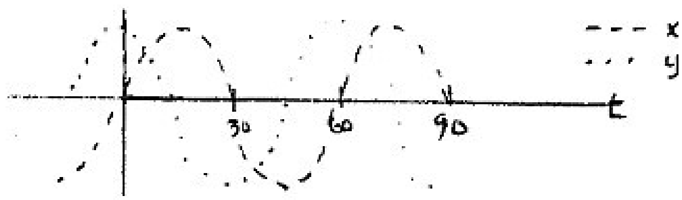

Figure 3-9. Sine and cosine superimposed.

Firstly, the cosine curve leads or precedes the sine curve by one quarter of a cycle, which is 90 degrees on the clock face. In the case of our gum, that quarter-cycle is fifteen seconds long. Secondly, the cosine curve is proportional to the *slope*, or steepness, of the sine curve. As the sine function becomes flat, the cosine becomes zero. As the sine function crosses zero and has its steepest upward climb, the cosine function is at its highest value. The sine function represents our upward view from the doorway; an oscillating motion from left to right and back again. The cosine function shows the perspective from a point on the wall beside the clock; a repeating up-and-down motion.

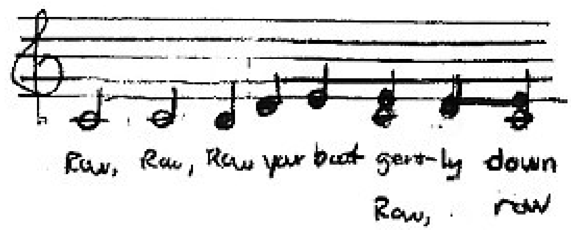

Figure 3-10. The relation between sine and cosine is like the relation of two parts of a musical round.

The relationship between sine and cosine is the same for any circular or helical motion and indeed for any sort of rotation. The variation between six and minus six inches from the clock's origin on the sine curve represents a rotation of 180 degrees. We have seen this rotational relationship already in the 180 degree difference between our perspective and Jane's: she counted six dollars and we counted minus six. What about the intermediate values between our perspective and Jane's? To someone standing off to the side and viewing the dollars from an oblique angle, each of the dollars would appear narrower and shrunken (perhaps reminding them of inflation). The width of the dollars would appear smaller and