Chapter 4
Perspective in Geometry: The Curved Earth

The Zen tradition includes many questions with difficult answers. For example, "What is the sound of one hand clapping?" The answer is not as important as the thought process it provokes. I will begin this chapter with a pair of Zen-like questions. Suppose you are at the north pole, the "true" north which is on the axis of the earth's spin, not the place where compass needles tend to point. Which way is (true) north from this North Pole? What date and time is it there? What time zone are you in?

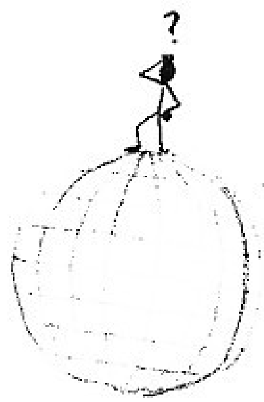

Figure 4-1. Which way is north from the north pole?

At the latitudes occupied by most people on Earth, the compass directions are fairly straightforward and are taken for granted. We also tend to think of them as continuous from place to place. If I start off in an eastward direction and keep going in a straight line, I would expect that I could travel long distances and find at each step of the way that I am still pointed east. We think of the compass directions not as local things, applicable only over short distances, but as global. Eastward is still eastward no matter how far you go. The *geometry* (from Greek roots reflecting the idea of "earth measurement") that we learn in high school tells us a number of rules. Two of these rules are that parallel lines never get closer or further apart, and that the sides of squares meet perpendicularly, meaning at 90 degree angles. These are the rules of *Euclidean* geometry, named after the ancient mathematician Euclid. On a flat surface, all of these rules are valid. They even work well over short distances on curved surfaces like the surface of the earth. But over large distances on curved surfaces, these rules do not hold. We call curved spaces "non-Euclidean" for this reason. How does the curvature of a surface affect the rules of geometry? How does the curvature of the earth affect our conventions of geography?

In the northern hemisphere, all of the night stars seem to circle around one particular star, and the direction of that star defines our north. The sun, moon, and stars all rise from a general eastward direction and set in the west. We have found that certain metals called magnets will tend to point to the north, and that discovery gave us the compass. Natural phenomena like these give us our sense of four directions.

These four directions can be described mathematically using a system established by the

French mathematician and philosopher Rene Descartes (1596-1650). A Cartesian coordinate system is all about rectangles and for this reason is also called the rectangular system. Pick any point on the map and make it your center, or “origin.” Draw a line running north-south through your origin and another going east-west. Each line is called an “axis.” From any other point on the map, draw a line from that point to the north-south axis and another to the east-west axis. In this way, every point on the map defines a rectangle having the two axes as sides. The rectangle's height is equal to its north-south distance from the origin and its width is equal to its east-west distance.

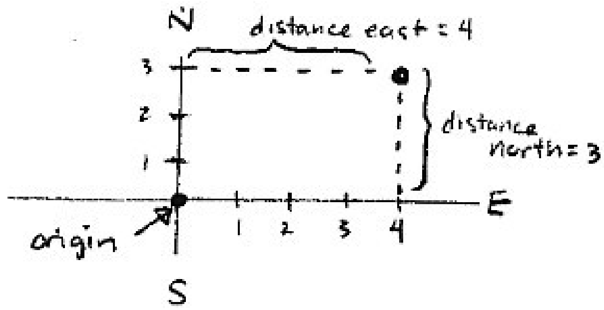

Figure 4-2. Measurement of Cartesian coordinates.

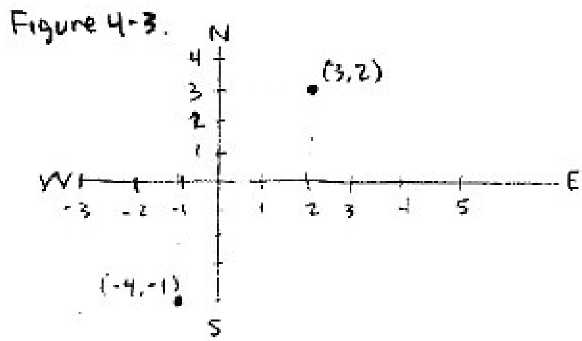

Figure 4-3. Notation system for coordinates.

Each point is also defined by two numbers representing not only the height and width of the rectangle, but the point's relative direction from the origin (north vs. south, east vs. west). For instance, a point 3 miles north and 2 miles east could be defined by the number pair (3,2). A point 4 miles south and 1 mile west of the origin would be defined by the number pair (-4,-1). These number pairs are called coordinates (see Figure 4-3). The two lines running through the origin are number lines as seen in chapter three. The beauty of the Cartesian coordinate system is that it combines geometry and algebra. By assigning numerical values to positions in multi-dimensional space, it makes the actual drawing of the map unnecessary. The mathematics can be done on the numbers alone. But as a grid of parallel lines, the Cartesian system depends on the assumption that space is flat; that parallel lines remain equidistant and do not converge or diverge. Geographically, we can treat the horizontal and vertical lines of the Cartesian grid as lines of latitude and longitude, but this only works in a limited way. If you pick up a globe, you will notice that the longitude lines running from north to south which are parallel at the equator actually converge at the north and south poles.

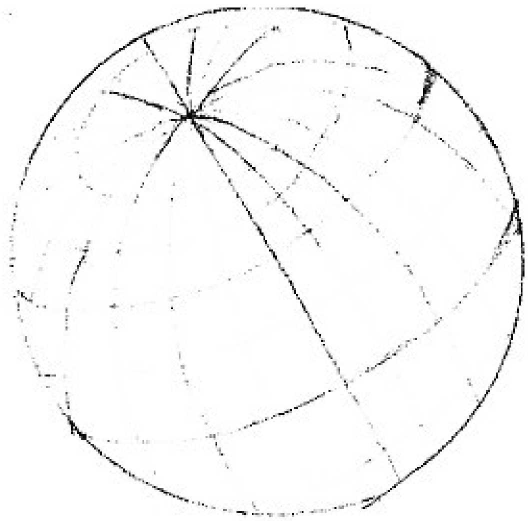

Figure 4-4. Lines of latitude and longitude.

An alternative to the Cartesian system is the polar coordinate system. Again, one designates a point on the map as the origin. But instead of two lines through the origin, only one axis is needed to establish direction. In this system, the two numbers needed to define any point are that point's *radial distance* from the origin and its direction from the origin (its *angular distance* from the axis), which can be measured by drawing a line to the origin and measuring the angle between this line and the axis. For example, a point 3 miles from the origin and lying on the axis could be defined by the coordinates (3,0). A point 3 miles away and in the opposite direction would be (3,180).

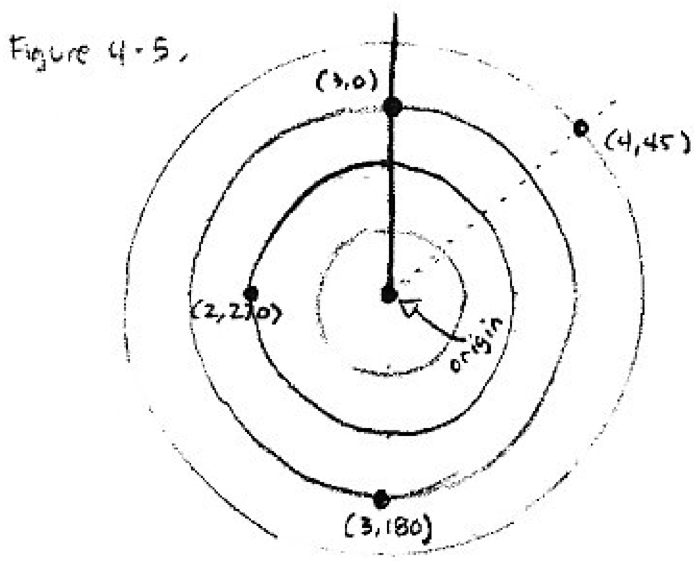

Figure 4-5. Polar coordinate system.

Again, one need not draw the map to calculate the distances between all three points. But this system also can only describe a flat space. Let's imagine that the origin is the north pole and that our axis is the Prime Meridian, or the line running between the north and south poles at zero degrees longitude. Let's draw another line running from the origin in any direction. On our flat polar coordinate system, this line diverges further and further from our axis and continues outward for ever and ever. But if we look at the same line on a globe, we find that at the equator this line actually runs parallel to the Prime Meridian and even meets it at the south pole. We can even follow it through the south pole where it continues onward, eventually meeting itself where it began.

The problem, then, is that for making a two-dimensional flat map of a two-dimensional curved surface, either of these coordinate systems only are valid locally – that is, over small distances – and not globally. This is similar to the problem of synchronizing the chiming of the clocks in chapter two; local synchronization is possible (in one place), but global synchronization is only approximate over small distances and completely impossible over large ones.

One coordinate system, Cartesian or polar, may be preferable to the other depending on the situation, but each has its limitations. On a curved surface, not only do parallel lines meet, but curved lines may remain equidistant. The perfect example of such curved lines are the lines of latitude on a globe. They run east and west and always remain the same distance apart. Why do they not meet as the longitude lines do? It is because the longitude lines are straight, but the latitude lines (except for the equator) all curve toward the nearest pole. They may appear straight, but if they truly were, they would eventually cross the equator, which is truly straight. Have you ever flown between North America and Europe to a destination which is more or less directly east or west, and noticed that your flight path curves northward and then south again? This is because your flight is taking a straight path which only appears curved on a flat map. This may be difficult to visualize, but pick up a globe and try it out. If you were two place two ends of a string on two points of equal latitude (other than the equator) and then pull it tight, the string would be at a different latitude in the middle. Still confused? Imagine two points one mile from each other in two opposite directions from the north pole. To go from one place to the other, you could go about one and a half miles directly east or one and a half miles west. Both directions from A to B would be equal. But the shortest path would be over the pole, exactly one mile. A related problem with flat maps of a curved earth is that what is square on a map is not square on the globe. The north-south lines are not parallel, and the corners do not all have equal angles. Various techniques in map making, such as allowing gaps or distortions on the map, are used for dealing with these problems.

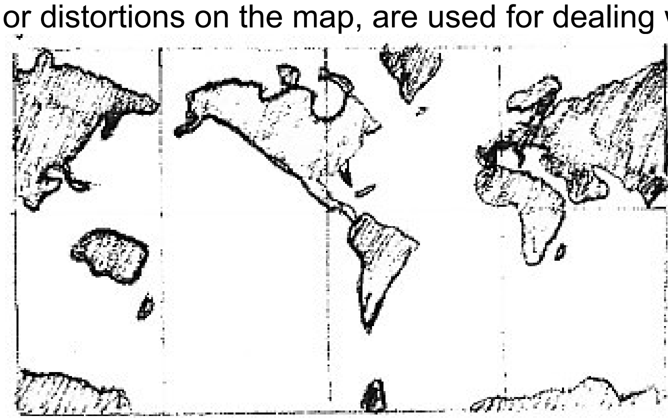

Figure 4-6. The Mercator projection tends to exaggerate the areas near the poles.

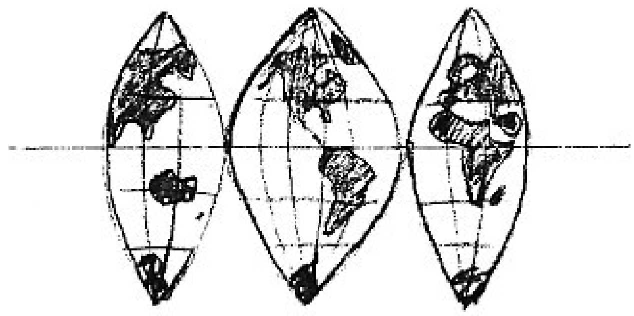

Figure 4-7. Some map projections reduce distortion by making the map discontinuous.

The definition of curvature is a bit slippery. According to the definition we will be using, a cylinder is not a curved surface; neither is a cone. The reason that these are not considered truly curved is that they can be made flat simply by cutting them. A sphere, on the other hand, cannot be made flat by cutting. It can be made to lie flatter by cutting, but each uncut area will still have *intrinsic* curvature. By contrast, we call the curvature of the cone or cylinder *extrinsic*. Parallel lines converge on the surface of a sphere. The interior angles of a triangle add up to more than 180 degrees. On the surface of a cylinder or cone, they do not. A sphere has what is called positive curvature. Both the inner and outer surfaces of the sphere have positive curvature. There is also a *negative* curvature in which parallel lines diverge, or get further apart from each other with distance from a given point; the interior angles of a triangle add up to less than 180 degrees. An example of a surface with negative curvature is a saddle or an hourglass shape.

The best way of mapping a two-dimensional curved surface is to make a three-dimensional map like a globe. The map can then be continuous and without distortions. A three-dimensional coordinate system is likewise better suited for describing the surface mathematically. For three-dimensional Cartesian coordinates, our north-south and east-west axes used above must be replaced by the more common, ambiguously-named lines called “x” and “y.” These lines are still perpendicular to each other, but since in three dimensions they could run past or directly through the earth, the four compass directions are no longer suitable. We also add a third “z” axis for our third dimension. The location of our origin and the orientation of our axes can be completely arbitrary, but for the sake of convenience let's put the origin at the center of the earth. The y axis will run between the north and south poles, and the x axis will run through the center at equator, meeting the Prime Meridian and the international date line. The x and y axes now form a plane which cuts the earth into two halves through the Prime Meridian and the international date line.

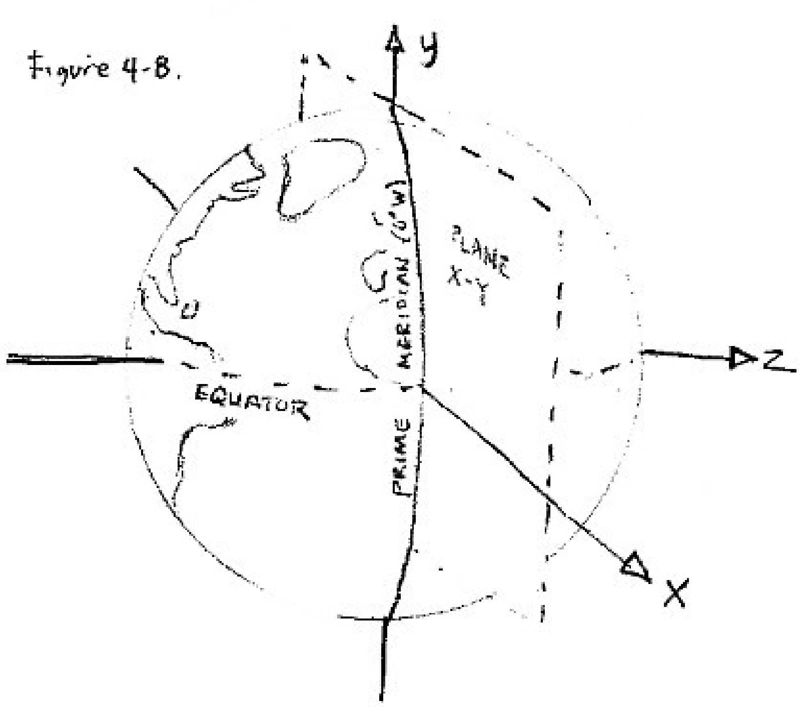

Figure 4-8.

The z axis must run perpendicular to these other two axes and the plane that they define. This origin and these three axes define one specific three-dimensional perspective with a reference point and three directions of orientation. Other perspectives can be defined by changing the origin or the orientation of the axes. But having defined one particular perspective, any point on, below, or above the surface of the earth can now be described by three numbers. If we define the radius of the earth as one unit of distance, then the xyz coordinates of the north pole would be (0,1,0). The south pole would be (0,-1,0). A spot off the west coast of Africa would be (1,0,0). Figuring the coordinates of most places, though, would require some trigonometry. Cartesian coordinates are easier to visualize for square volumes than for spherical ones. If we put the origin of our coordinate system on the floor in the corner of a room and oriented the axes along the lines where the walls and floor meet, then any point in the room can be described by three easy-to-measure numbers: Its distance from one wall, its distance from the adjacent wall, and its height from the floor. Instead of the *rectangle* defined by any given point in a two-dimensional Cartesian system, a point in three dimensions defines a *cube*, with the origin occupying one corner and described point occupying the opposite corner.

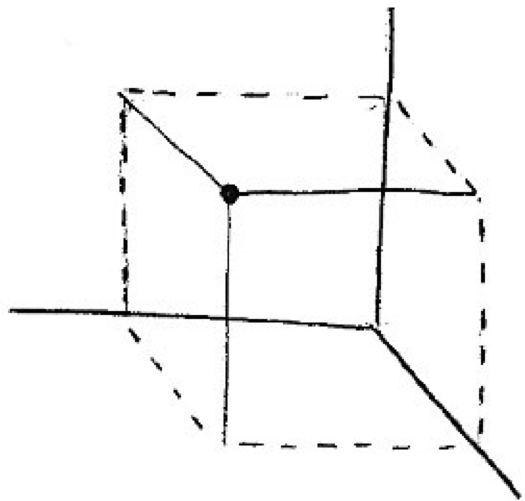

Figure 4-9.

Cubes are easier to visualize as lining up in the corner of a room than as touching the center of the earth. For describing points in, on, or near the earth, a three-dimensional *polar* coordinate system is more useful and is indeed what we use most commonly. Just as in the example above, we choose the center of the earth as our origin and the axis of earth's rotation as one axis. The other axis runs between the international date line and the Prime Meridian at the equator. Any point we occupy can then be defined by three numbers which correspond to the two polar coordinates in our earlier two-dimensional example. We still have one number for distance from the origin, but instead of measuring one angle, we measure two. We also fudge the numbers a bit and count backwards, but the basic idea is the same. We call our three numbers latitude, longitude, and altitude. Altitude is our height above (or below) mean sea level, and really is our distance from the center of the earth, or the origin. We just subtract the depth of the earth for simplicity's sake. Our latitude is defined by imagining a line from our position to the center of the earth and then measuring the angle between this line and the north-south axis. In a typical polar system, this angle would be our second coordinate. The north pole would be zero, the equator 90, the south pole 180. But for latitude, we adjust the measurement a little but. There are still 180 degrees of latitude, but we assign zero to the equator, 90 to the north pole, and -90 to the south pole. In common parlance, though, instead of saying “minus 90 degrees north,” we say “90 degrees south.” Our third number is longitude. This is measured by finding the point nearest to us on the equator, drawing a line from there to the center of the earth, and measuring the angle between this line and our second axis. At Greenwich and all other points directly north and south of it, this angle is zero. This angle can vary 180 degrees east or west.

I want to point out that any origin, any orientation, and either three-dimensional coordinate system will enable us to measure places and distances in three-dimensional space. But one coordinate system may be more convenient than another in some particular problem, and one origin or orientation may be more convenient than another. Let's go back to the clock tower example from chapter two These clocks were arranged in a grid with 1000-foot squares. We could pick the tower on the southwest corner as the origin and line up the x and y axes to point north and east. The origin clock would be at (0,0). If we count 1000 feet as a single unit of distance, then all 100 towers have whole-number coordinates, with the x and y coordinates for the whole system ranging from 0 to 9.

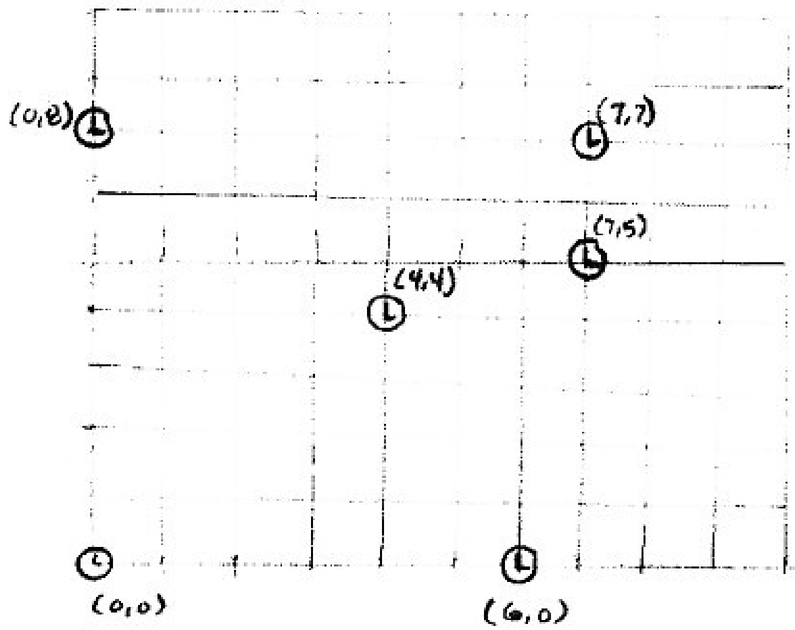

Figure 4-10. The coordinates of various clocks.

A polar coordinate for each clock might require more calculation. Using the same origin and the same unit of distance, the clocks lying directly north or east of the origin would each have a whole-number distance coordinate, but most of the others would be measured with decimal values which we could calculate using trigonometry. The angle coordinate would likewise require some calculation. So the Cartesian system seems more appropriate for noting the positions of the clock towers. Since the spacing of the towers is uniform throughout the system, we can say that it is *global*. A Cartesian coordinate system is often more useful for global phenomena. However, if we are permanently located in one tower and we want to predict which towers we will hear at given times, or which towers will hear us first, then we may find it useful to keep a list of polar coordinates for the other towers, placing ourselves at the origin. The first polar coordinate is the distance coordinate. The towers with the smallest distance coordinate will hear us first. As we saw in chapter two, the apparent time order of the tower bells is a *local* phenomenon. A polar coordinate system is often more useful for local phenomena, or problems in which radius figures prominently. This distinction is important to make, because we have seen and will continue to see in later chapters that time can only be measured locally, not globally. We often consider time to be equal to space as another Cartesian axis of measurement (as in our light cone diagrams), but this is a mistake. Time is local, and is therefore more easily represented in polar coordinates; it is more accurately conceived not as running along any axis but as depending on the distance from the observer, as with our clock towers and race officials.

Let's suppose for a moment that political boundaries do not affect the layout of the earth's time zones, and that all time zone lines run directly north and south at every fifteen degrees of longitude instead of zig-zagging around political borders. Likewise, we'll straighten out the international date line to lie on 180 degrees longitude. Let's stand at the north pole and watch how dates and times change throughout the day:

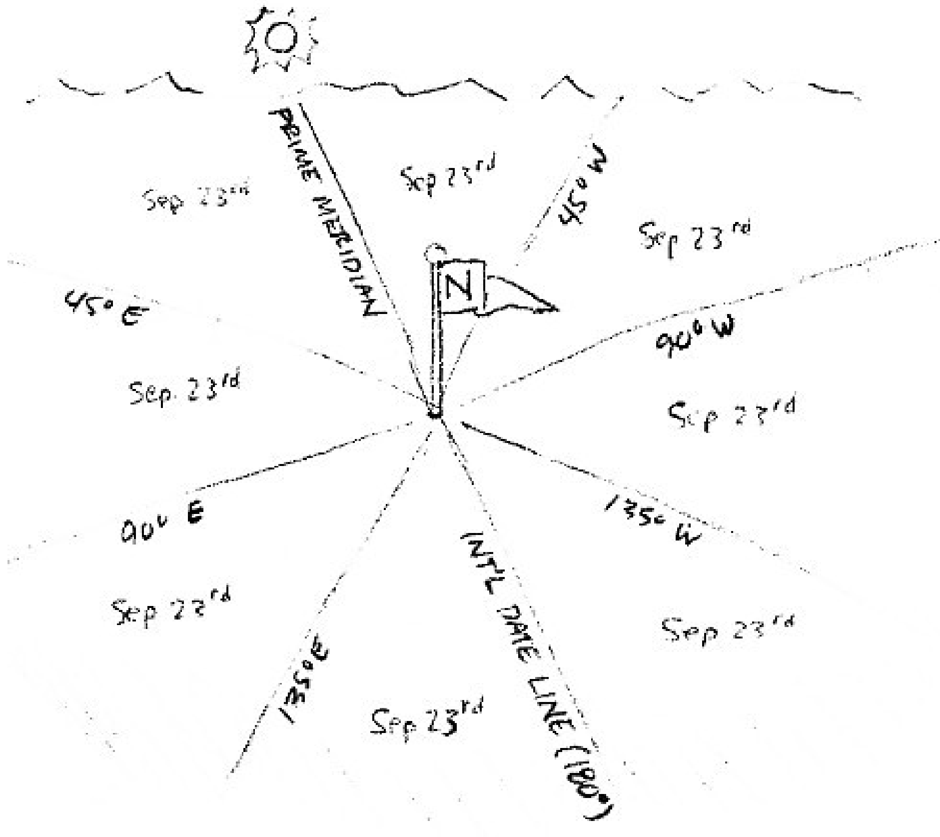

Figure 4-11.

When we start out, it is September 23rd everywhere on earth (Figure 4-11). The sun is on our horizon, directly over the Prime Meridian. Over the next 24 hours, we watch the sun scrape over the horizon from left to right. From overhead, this motion would appear clockwise. In fact, the sun is moving around the horizon like an hour hand over the numbers on the face of a clock. But our clock is numbered from 1 to 24. The sun starts over the 12 and as it moves, the time zone on the opposite side of the clock enters September 24th. As one side of the globe passes from 11:59 (a.m.) to noon, the opposite side passes from 23:59 (11:59 p.m.) to midnight and into the next day (Figure 4-12).

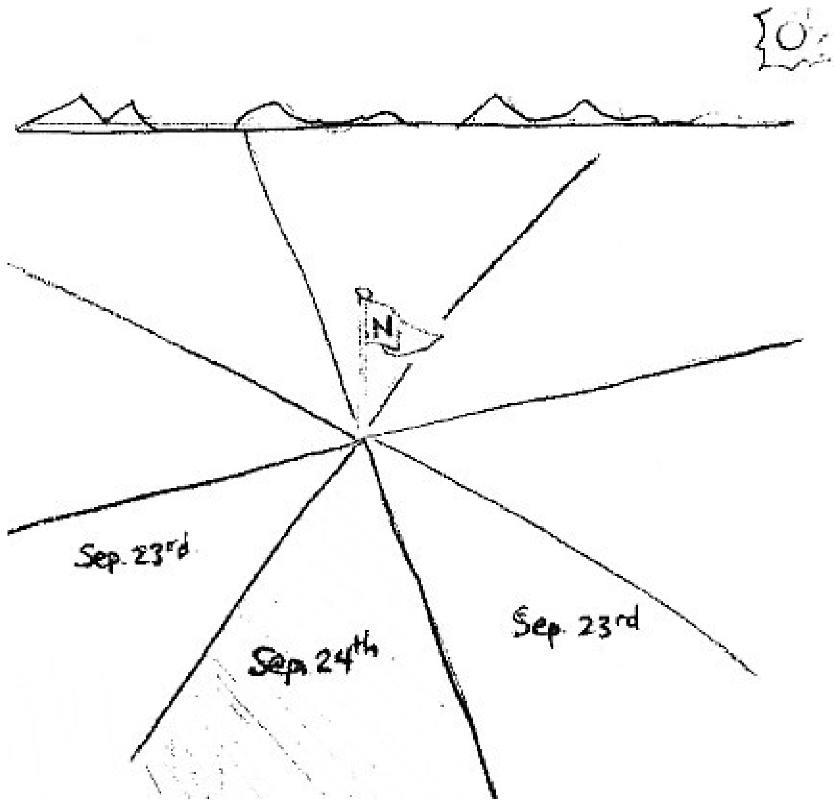

Figure 4-12.

Over the course of the sun's journey, the pie-shaped slice of our world that is September 24th grows, while the rest of the pie shrinks. Eventually September 24th encompasses the whole globe and the cycle begins again.

So now we return to the questions we began the chapter with. What direction is north at the north pole? The answer to that question is, “Here” or “This way.” North is not so much a direction as a location. What time of day is it at the north pole? This question can be approached several ways. One could say that the date and time are undefined. One could also say that it depends on which direction one is facing. Toward the sun is noon. The opposite direction is midnight, and so on. At the north pole, time is not a location but a direction. In a very abstract sense, time and one dimension of space switch places.