Chapter 9
Perspective and Motion: Electromagnetism

James Clerk Maxwell might be considered the last great scientist of the classical age or the first of the modern age. His study of thermodynamics contributed to the development of quantum theory. His assembling of the four equations defining electric and magnetic fields and the relationship between them explained the nature of light; this set Einstein on the path to relativity. The majesty and ambition of Maxwell's Equations call to mind a passage from Beethoven's 9^{th} Symphony: “Drunk with fire, O Heavenly One, we enter thy sanctuary. Thy magic again joins what custom has strictly divided (translation by author).” The equations unite electricity with magnetism and the poetry of science with the music of mathematics. They are the mathematical expression of God's decree, “Let there be light.”

Electromagnetism was discovered by Hans Oersted in 1820; he noticed a compass needle jump whenever a nearby electric circuit was switched on or off. The interaction of electricity and magnetism can also be demonstrated in other ways. As mentioned already, current flowing in a straight wire will create an apolar magnetic field around the wire (Figure 6-11). Circular current, or current through a coil, will produce a polar magnetic field (Figure 6-12).

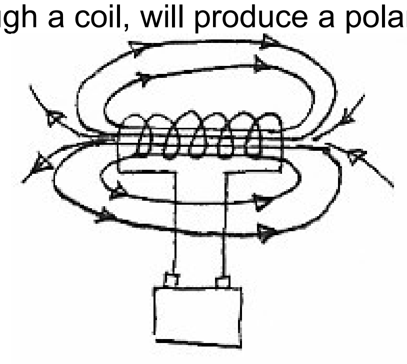

Figure 9-1. Induction of polar magnetic field using helical current.

Electric generators turn spinning magnets into current, and electric motors do the reverse. Bundled wires may attract or repel depending on whether the direction of the current flow in them is equal or opposite. A spinning charged body will generate a magnetic field having poles aligned with its axis of spin. A charge moving in a magnetic field will be deflected in a direction perpendicular to its movement and to the magnetic lines of force. The variations of these interactions are at first mind-boggling, but they are all encompassed in Maxwell's four equations, which I will introduce and then attempt to translate into less concise but more readily understood language:

Equation 9-1. $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$

Equation 9-2. $\nabla \cdot B = 0$

Equation 9-3. $$\nabla \times E = -\frac{\partial B}{\partial t}$$

Equation 9-4. $$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

Don't panic! Those upside-down triangles (nabla symbols) are used to represent divergence and curl, which we have already learned about (and which are explained in greater depth in Appendix B). The rest can be explained in everyday terms too. Let's take these equations one by one.

1. $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ This formula is also known as Gauss' law. It states that in any space, the divergence (symbolized by the nabla and dot) of the electric field, *E*, is proportional to the charge density (the Greek letter ρ or "rho") in that space. You may recall that electric charge is attributed to particles called electrons and protons. Put simply, this law states that the number of lines of electrostatic force radiating from a space is proportional to the net difference in protons and electrons in that space. An overbalance of electrons will create a net negative charge and will be represented by a proportional number of lines pointing into the space; an equal number of protons and electrons will result in no net charge, and no lines of force will be shown; an overbalance of protons will create a net positive charge and will be represented by lines pointing out of the space. As is our custom, we'll ignore the constant (ε) in the equation.

2. $\nabla \cdot B = 0$ The second equation states that the divergence of a magnetic field, *B*, is zero. In other words, the number of lines of magnetic force leaving any space is equal to the number of lines entering it from the other side (Look at Figures 6-9, 6-12, and 9-1 for examples). The number of lines radiating outward and not returning is zero. For this reason, there is no such thing as a magnetic monopole; every north pole is a south pole when you turn it around.

The first two laws define the electric and magnetic fields. The final two laws define the relationships between them, and are a little more difficult to visualize. These laws state that each field is affected by changes in the magnitude or direction of the other. The nabla and multiplication symbols together represent curl. It is helpful to think of the left-hand sides of these equations as representing circular lines of force.

3. $\nabla \times E = -\frac{\partial B}{\partial t}$ This law is also known as Faraday's law of induction. Though magnetic lines of force can never be made to diverge (to be one-directional and straight), electrical lines of force can be made to curl, that is, to form a closed loop, or a

circular difference in voltage. If an electrical conductor exists all the way around this loop, current will flow through it and we will have an electrical circuit. The direction and magnitude of the voltage in the circuit is represented in our third equation by the curl of the electric field *E*. The direction and magnitude of the magnetic field is is represented on the other side of the equation by the vector *B*. The $\frac{\partial}{\partial t}$ (read "partial d over partial dt") in the equation represents change over time, specifically the change in *B* over time (for a more in-depth explanation, see Appendix C). If the strength of this magnetic field changes (right side of the equation), so does the voltage in the circuit (left side). Some flashlights can be powered by the up-and-down shaking of a magnet through an electrical coil. As the magnet enters and exits the interior of the coil, the strength of the magnetic field around the coil rises and falls; this creates voltage first in one direction and then the other. This follows what is known as the right-hand rule. Picture the fingers of your right hand as the coil and your thumb as the magnet (the tip of your thumb being the south pole). If the strength of the magnetic field decreases in the direction of your thumb (the lines of magnetic force point in the opposite direction of your thumb) because you pull the magnet out of the coil in the direction of your thumb, a voltage will be generated in the direction of your fingers until the magnet is out and the field strength falls to zero. As you put the magnet back in again, voltage occurs in the other direction.

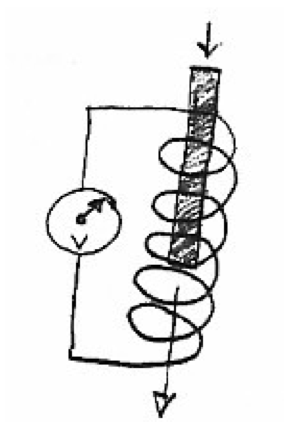

Figure 9-2. A magnet drops through a coil, inducing current.

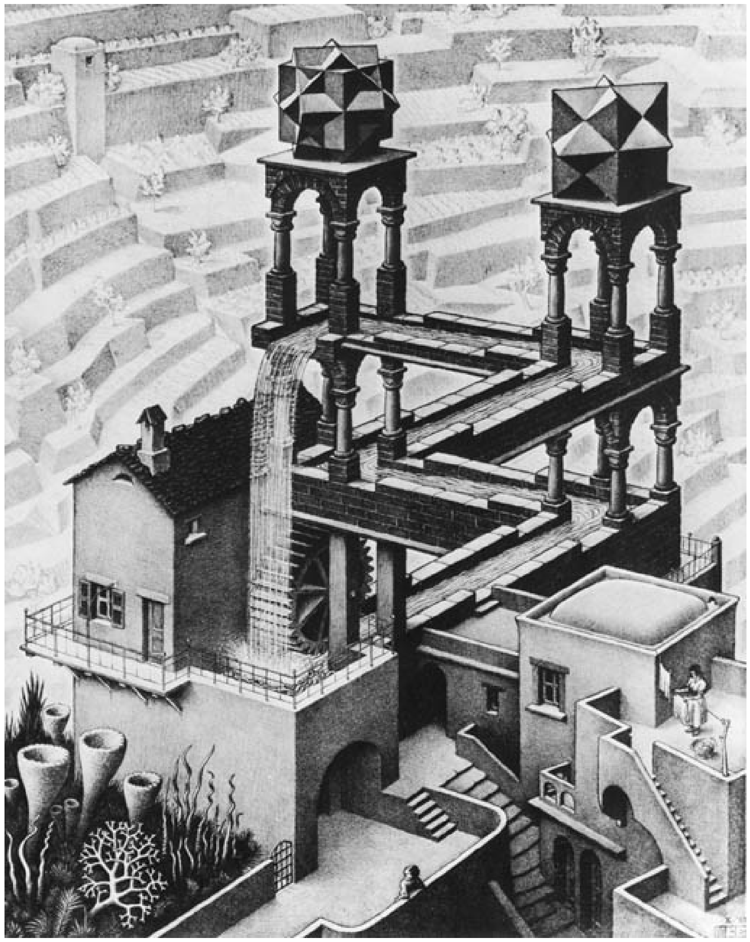

Figure 9-3. Escher's waterfall. Like water, electricity flows from higher to lower potential. A changing magnetic field can make electric charges flow in a circle.

Voltage can also be changed by a change in the magnet's orientation. If a magnet rotates near a coil (on an axis other than the polarity of the magnet), the magnetic field around the coil will both rise and fall, reversing directions as it goes, creating voltage first in one direction and then the other. The amount of voltage in the coil or circuit caused by the movement of a magnet is also proportional to the number of turns in the coil (think of the number of fingers on your hand). As the magnet either rotates or moves back and forth, the voltage will have a sinusoidal graph over time and will create what is called alternating current.

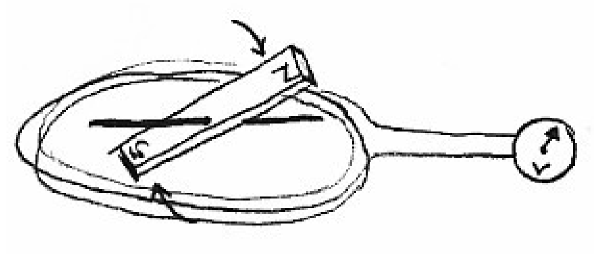

Figure 9-4. Rotating magnet in coil.

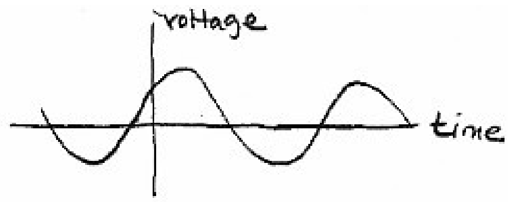

Figure 9-5. Voltage variation in an AC circuit.

4. $\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$ Lastly, we have the equation stating that magnetism is created both by electric current and by changes in voltage. When we look at this together with the second equation, $\nabla \cdot B = 0$, we discover that magnetism has no independent existence; the sole bases of magnetic force are electric current (J) and changes in electrical force ($\frac{\partial E}{\partial t}$). This is an important realization to come to. All magnetism arises from electric current or changes in voltage, which is the direction or strength of the electrical field. As current flows in a wire, lines of magnetic force create closed loops around it. The direction and density of the current flow, *J*, determine the direction and intensity of the magnetic field. This follows the right-hand rule: If current flows in the direction of your thumb, the lines of magnetic force will encircle the wire in a counterclockwise direction. If the wire is straight, the field is apolar; that is, it will have no north or south pole (Figure 6-11). But if we bend the wire into a circle or coil, magnetic poles are formed on an axis perpendicular to the plane of the coil (Figure 9-1). A long current-carrying coil has a magnetic field just like a bar magnet. In fact, if we place an iron nail within this coil, it will slowly become magnetized and remain so even after the current stops flowing. Our fourth law explains what magnetism in metal

actually is: an alignment of circular currents or voltages in the metal. In unmagnetized metal, electrons revolve around the individual atoms in the metal in random directions. These circular currents create tiny, random magnetic fields which cancel one another out on the average. When the orbits of some of the electrons become aligned and the randomness decreases, an overall magnetic field is measured. The last term in the fourth equation states that changes in the electrical field over time ($\frac{\partial E}{\partial t}$) create changes in the magnetic field.

Maxwell realized that these equations defined a bi-directional relationship between magnetism and electricity. A changing electrical field would create a changing magnetic field, which in turn would create another changing electrical field, and so on indefinitely. A change in either field would create a self-sustaining chain of processes which would radiate outwards as an *electromagnetic wave*. It was only after Maxwell calculated the speed of this theoretical wave that he realized he had created an explanation of light. In the following years, other scientists were able to create and detect electromagnetic waves of lower frequencies than those of light. Visible light, infrared light, radio waves, microwaves and x-rays are all forms of electromagnetic radiation.

Notice that there is no $\frac{\partial}{\partial t}$ in either of the first two equations, the equations defining divergence. No changes in any field will create static charge (a divergent electric field), nor is there any way at all to have a magnetic monopole (a divergent magnetic field). The idea of change, $\frac{\partial}{\partial t}$, only appears in the equations for curl. Motion can create a static magnetic dipole or a circular voltage. We found the same thing in our freeway experiment. No divergence was created by our motion, but we were able to both create curl and define its direction by our motion. In our wide-angle perspective drawings of the railroad tracks, our motion down the tracks appears to create a circular current of railroad ties. When we stop, the current stops.

One of the more perplexing interactions of electricity and magnetism is the way the motion of a charged particle can be diverted by a magnetic field. I will give three examples which all exhibit this strange behavior. First, Let's stand an imaginary bar magnet on end with the south pole at the top. Now we'll shoot negatively-charged particles at it. All of the particles get diverted to the right and miss their target, without losing speed. The angle of their deflection depends on their velocity, mass and charge, and on the strength of the magnet.

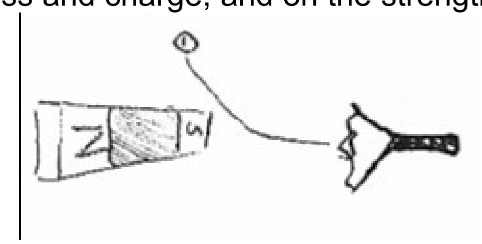

Figure 9-6. Bar magnet as target for electron slingshot.

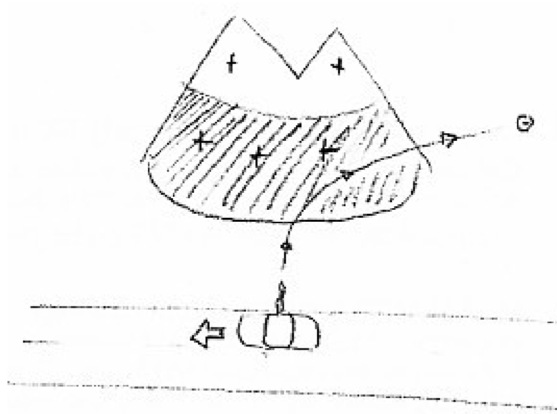

Figure 9-7. Charged mountain as target for moving electron slingshot.

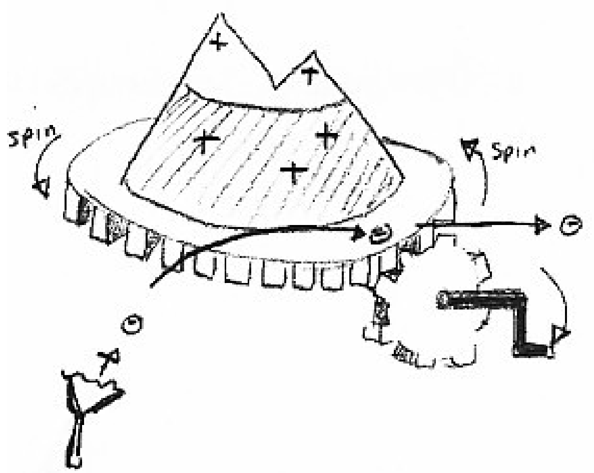

Figure 9-8. Spinning, charged mountain as target for stationary electron slingshot.

For our second and third examples, let's go back to the freeway. Directly out the right-hand window on the other side of the plowed field is Little Mountain, which somehow we have managed to imbue with a large positive charge. We are traveling at freeway speed and we shoot some positive charges at it. Again, the charges are diverted to the right and miss the target. Now we stop the car and – through some marvelous feat of engineering – we pull a lever and cause the mountain to rotate in place. We shoot more positive charges at it, and again they are deflected to the right. All three experiments produce the same result because they all have equivalence. In the first case we have a magnet, which we know to be no more than a collection of rotating charges with non-random alignment. In the third case, we cause the charges in the mountain to rotate, effectively inducing it to gain a magnetic field. The second case is less obvious. Our motion relative to the mountain gives it an angular velocity in our frame of reference. This situation is equivalent to it rotating. As we established in chapter eight, the angular velocity of the mountain directly out our side window is inversely proportional to the square of its distance from our car. Since the angular velocity is the source of the magnetic field, we should not be surprised to find that the strength of the magnetic field is inversely proportional to the square of our distance from it. As our charged particles fly downrange toward the mountain, they retain the same linear velocity that our car has on the freeway. As they get closer to the mountain, the angular velocity of the mountain with respect to them increases exponentially, and therefore the magnetic field strength due to the apparent rotation of the mountainous charge increases at the same rate. The equivalence of our motion past the mountain and the rotation of the mountain is important to note. In both cases, there is an angular velocity at work *in our frame of reference*. Not all observers will measure magnetic force from the charged mountain. In the second example, we only measure magnetic force because we are in relative motion on the freeway. Someone standing in the field will not measure the magnetic force. The mountain does not become magnetized just because we are moving past it. It doesn't notice us and does not care. We create the magnetic force by our own motion. In the third example, we measure magnetic force from the charged mountain because we see it rotating on whatever massive machinery it is connected to. But anyone who is on the mountain will not notice any magnetism from the charged mountain. They will not find, for instance, that metal objects in their pockets are pulled toward the ground. From their frame of reference, the mountain is not rotating.

That gives us our quantitative, mathematical explanation. We know how much the charges from our gun will be diverted. But we still haven't discussed the qualitative explanation, the reason why. Why do the charges get diverted sideways and why do they maintain the same speed in the process? To understand why this happens, we need to understand angular momentum. The behavior of a gyroscope can be every bit as astonishing as the behavior of charged particles in a magnetic field, and the same principle is at work. Here is an experiment you can easily do at home. Pop the front wheel off of your bicycle and hold it by the axle. Notice that the axle and wheel are lubricated so that you can spin the wheel freely while holding the axle, and very little of the wheel's angular momentum will be lost to friction. Now hang the wheel from some sort of line by tying one end to the axle. With the line tight, tilt the wheel until the axle is parallel to the ground. Let the wheel go and you should not be surprised to see it tip over until the free end of the axle points at the floor. Now tilt the wheel again so the axle is again level with the ground. Take hold of the rim and give it a really good spin, the faster the better. Let it go. If the wheel is spinning fast enough, it will only tip over slightly. Instead of falling over, it will start to spin on a vertical axis, getting a little faster over time. Eventually, as the wheel loses the spin around its axle, it will tip over more and more. How is it that the wheel appears to temporarily defy gravity, not tipping over when only one end of the axle is supported?

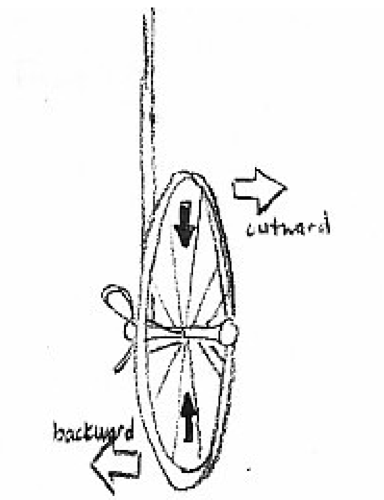

Figure 9-9. A (non-spinning) bicycle wheel hanging from a string tied on one end of the axle. Gravity causes the wheel to tip over.

Let's take a closer look at the non-spinning wheel. When we tilt it up and let go, there are two forces at work on the wheel. One force is due to gravity operating on the entire wheel and this force is downward. The other force is the upward force from the line. Since this force only acts on one end of the axle, it creates *torque* on the axle, which is to say that it causes the axle to rotate from its horizontal direction toward a more vertical one. In Figure 9-9, an upward force on the rear end of the axle causes the top of the wheel to move outward and the bottom to move back. These are *linear* forces, indicated by the white arrows. But do these points of the wheel continue in these directions? No, they accelerate continuously toward the center of the wheel and keep a circular path. There is a force at work within the wheel, a *radial* force toward the center of the wheel (black arrows). This is the force of the spokes pulling inward on the wheel and keeping them from continuing in a straight line. With two forces at work, the top of the wheel comes outward and down; the bottom moves backward and up. This is not very complicated, but what happens once the wheel is spinning? While the linear momentum gained from the torque keeps its direction, the radial force on each point of the wheel *changes directions*. The top was being pulled down by the spokes, but soon it is being pulled to the right. The bottom was being pulled upward, but soon it is being pulled to the left. The momentum acquired from the original torque, combined with the changing radial force, creates a *new* torque which is in a direction 90 degrees from the original. The upward force on one end of the axle causes that end of the axle to accelerate to the left. Or you could say that the downward force on the other end causes it to accelerate to the right. Each end accelerates in an opposite direction, and this causes the wheel to spin on a vertical axis.

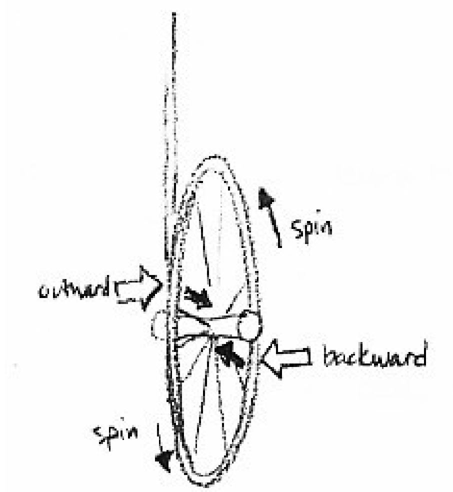

Figure 9-10. A spinning bicycle wheel. The spin translates gravity's pull into a different direction.

Now, to answer our earlier question. Is the wheel temporarily defying gravity? No, it *tries* to tilt downward, but keeps missing. It is the same reason that satellites can stay in orbit. They are accelerated toward the earth by gravity, but due to their high orbital speed, they keep missing it.

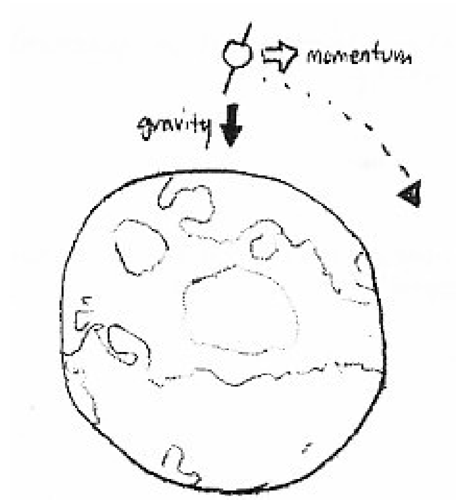

Figure 9-11. Satellites fall toward the ground but constantly overshoot it.

What does angular momentum have to do with the forces acting on charged particles moving in a magnetic field? Everything. Let's set up an imaginary lab with two charged masses, a ball and a pipe. We'll give each a negative charge. We throw the ball at the pipe and note that it is repelled. We may be able to hit the pipe, but the ball does not hit as hard as it might have because the pipe and the ball are pushing back at each other due to their like charges. The movement of the ball when we throw it creates a magnetic field around it, but there is no other magnetic field for it to interact with, so it does not matter to us at this point.

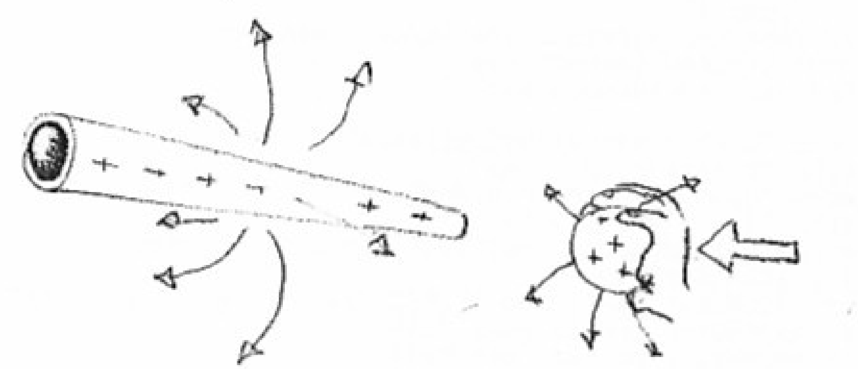

Figure 9-12. Ball approaching pipe of like charge. The lines of electrical force around each object seem to be hinting that the objects would prefer to go any direction than next to each other.

Next we attach the pipe to a motor and make it rotate. This causes the all of the charges in the pipe to spin in parallel circles, and since the pipe carries a net charge, spinning it creates a polar magnetic field (Figure 9-13). We throw the ball at the pipe again, and the negative charges in the ball will still push on the charges in the pipe. But since the charges in the pipe are now rotating, the results will be different. The movement of the charged ball creates an apolar magnetic field. The charged ball is moving perpendicular to the pipe's magnetic lines of force, so its magnetic lines of force will create torque on the dashed lines, on the axes of all those spinning charges. As we have already seen, torque on such an axis results in a new force having a direction perpendicular to both the torque and the axis. This means that in addition to pushing back on the ball, the pipe will push either left or right on the ball, depending on the polarity of the ball and the pipe. Next we disconnect the pipe from the motor, coil heavy wire around it, and run current through the wire. This causes the negative charges in the pipe to spin in tiny parallel circles, and the pipe thus becomes magnetized. The effect of the magnetized pipe on our thrown ball is the same as that of the rotating pipe.

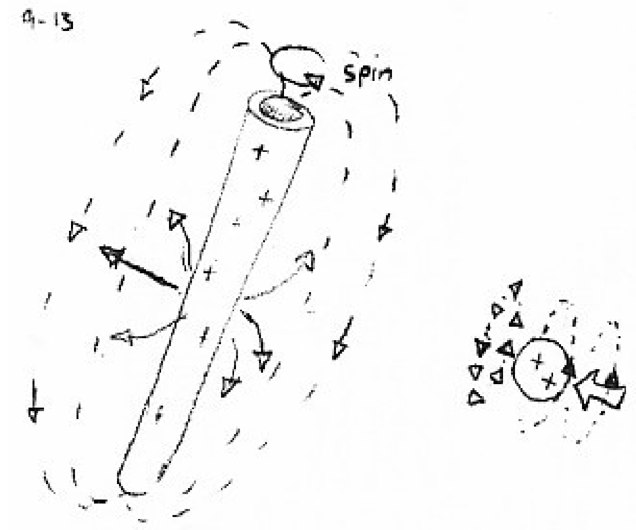

Figure 9-13. Ball approaching spinning or magnetized pole of like charge.

I have described several situations, some of which are equivalent to one another even though they appear somewhat different. Equivalence is one of the key concepts of relativity. Relativity says that when there is no background to measure against, there can be no distinction between being in uniform motion and being at rest; these two states are equivalent. To say that you are in motion is meaningless unless you specify, “I am in motion relative to the ground.” When several observers are in uniform motion but have no other standard to measure against, any of them can base their coordinate system on the assumption that they are at rest and the others are moving, and their measurements will match the expected laws of physics. But none of them can say that – since they assume themselves to be at rest and the others to be in motion – they are any sort of special case. Each of their situations are equivalent. If any of the observers collide, everyone will agree that momentum was conserved in the collision, even though they may disagree on the velocities involved.

The same equivalence applies to relative motion between a bar magnet and a surrounding coil of wire. Whether we say the magnet is at rest and the coil is moving, or vice versa, or that both are in motion, we will still measure the same voltage in the coil as long as the difference between the magnet's motion and the coil's has the same mathematical value. The principle of equivalence extends to angular motion as well as linear motion. Whether we are running in a circle around a charged body, running in a straight line past it, or at rest while the entire body rotates, or at rest while the individual charges in the body rotate in alignment, or if *both* we and the charged body are in motion, we will measure a magnetic field around that body in direct proportion to the difference in angular velocity between us and its charges.

Our final topic in this chapter is the exotic-sounding concept of *imaginary numbers*. To the mathematical and scientific dilettante (such as myself), imaginary numbers may seem like an amusing but ultimately useless idea. My introduction to the idea of imaginary numbers was in high school algebra and connected to the concept of the square root. The square root of a particular number, as you may recall, can be multiplied by itself to result in that particular number. The square root of nine is three; of four is two; of one is one. Each number actually has two square roots, a positive root and a negative root. Minus two times minus two is also four. Minus one times minus one is one. Squaring any (real) number always gives a positive result. So one might ask, being particularly curious or impractical, what is the square root of minus one, or of any negative number? An answer to this question was proposed in the sixteenth century. If we define some number as the square root of minus one, the square roots of any other negative numbers will simply be multiples of that number. Today we call this magic number *i*, the imaginary number. "i" squared is minus one, and the square root of minus one is i. The square root of minus four is 2i. 3i squared is 9. "That's fine," you might think, "but when will I ever encounter a negative number and need to know its square root? Isn't this just one of those things that mathematicians come up with to impress each other?"

Imaginary numbers alone may seem a useless abstraction, but put together with those numbers that we call "real," they are part of very useful "teams" called "complex numbers." A complex number can be expressed as the sum of two numbers, one of them real and one imaginary. For example, the complex number "4 + 2i" has a real part (4) and an imaginary part (2i). The real number 8 could be expressed as the complex number 8 + 0i. Complex numbers are useful in describing wave-like phenomena. Alternating current (AC) is a good example. Let's set up a coil which is in the presence of a rotating magnet. The whole arrangement goes into a closed black box, but the ends of the coil poke out of the box and we can measure the voltage difference between the two ends. We will notice that the voltage peaks and reverses direction over time, like this:

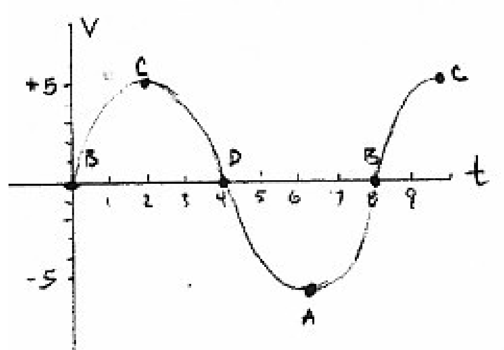

Figure 9-14. Voltage in an AC circuit.

This circuit reaches peaks of plus and minus 5 volts. The voltage *v* at any given time *t* can be expressed as a single real number. For instance, at *t=1* the voltage is 3.5 volts. But this single number only tells us the voltage at that time, not whether the voltage is increasing or decreasing at that moment. By adding an imaginary term and using a complex number, though, all of this information can be included. Some clever fellow, I don't know who, figured out that the slope of the voltage graph at any moment can be expressed as an imaginary number and that when added with the instantaneous voltage, the resulting complex number encodes both the instantaneous voltage and its *phase* (among other things, whether the voltage is increasing or decreasing).

Let's create a two-dimensional graph of "real" and "imaginary" voltage without respect to time.

Real voltage will be on the horizontal line and imaginary voltage (or the rate of increase of real voltage) will be on the vertical line. First, we notice that voltage is at its high and low peaks when its rate of increase is zero (the graph in Figure 9-14 flattens out at the peaks, points A and C). So we'll make two marks (points A and C on Figure 9-15) for the high and low voltages on the horizontal axis, where the “imaginary voltage” is zero.

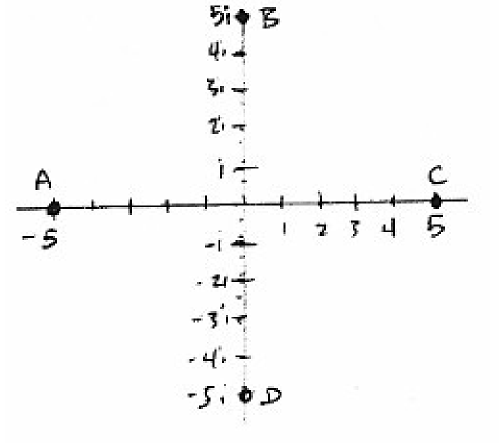

Figure 9-15.

Next we graph the two points where the voltage is zero; as we see in Figure 9-14, the slope of the voltage graph is sharpest at these times (points labeled B and D). At t=0, voltage is zero but sloping upward; at t=4, voltage is zero but sloping downward. So we create two marks on the vertical line where voltage is zero but slope is either positive or negative. If you'll take my word for it that the maximum slopes of the voltage in Figure 9-14 are 5 and minus 5, we get points B and D on Figure 9-15.

Since Figure 9-14 is a continuous graph with no gaps or sharp turns, we might expect that all of the points in Figure 9-15 would connect in a manner just as smooth. Indeed, each of the points in Figure 9-14 can be translated into a circular graph of voltage versus rate of change in voltage:

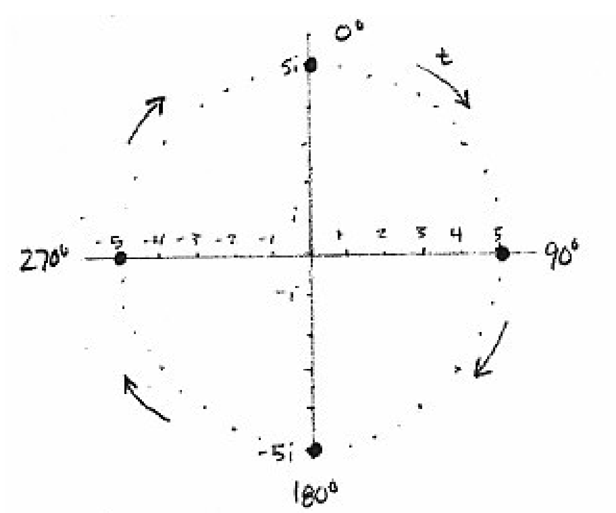

Figure 9-16.

Depending on the frequency of the voltage change, the v-to-i graph will be circular (as in Figure 9-16) or elliptical. For high frequencies, the voltage changes rapidly and thus the rate of voltage change (the imaginary term) will have a broader range. For low frequencies, on the other hand, the imaginary term will have a narrower range. Now we come to the part where we assign units to our graph. The frequency of the alternating voltage in Figure 9-14 is such that in the instant the voltage is decreasing to zero, it is decreasing at a rate of five volts per second. Since the peak voltage is five volts, the v-to-i graph is a perfect circle. For a higher frequency alternation in the voltage, the slope of the voltage vs. time graph would increase, and the v-to-i graph would be taller:

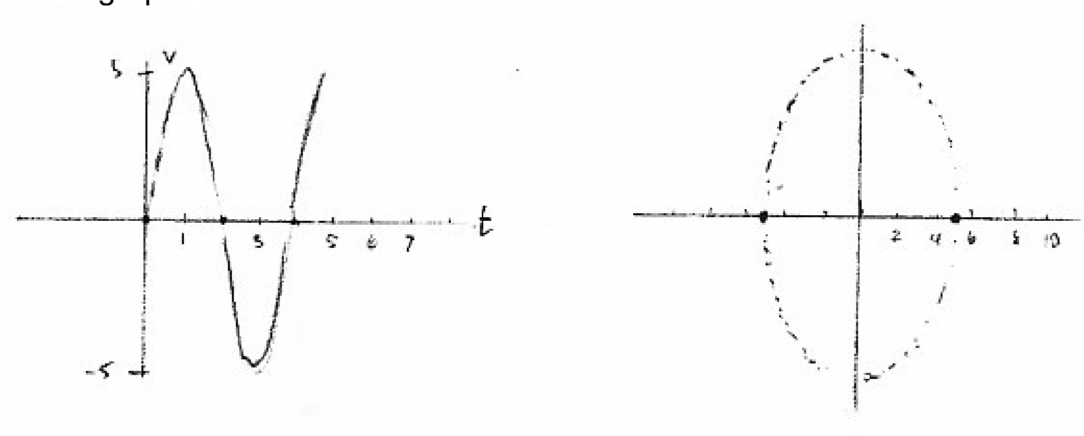

Figure 9-17.

For a lower frequency, the v-to-i graph would be flattened:

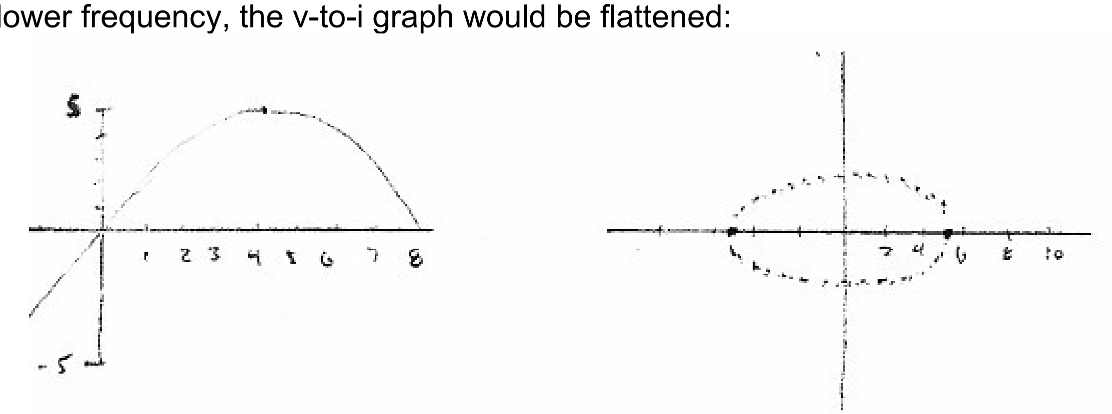

Figure 9-18. Notice that the ellipse here appears the same as if the circle in Figure 9-16 had been rotated on the horizontal axis.

Now we can imagine that instead of a rising and falling wave, the voltage makes a clockwise circle on our graph of *v* versus *i*. Whereas in Figure 9-14 we had a graph of the voltage at any given time, in Figure 9-16 we have a graph of the voltage (v) and its rate of change (v/t) ("imaginary voltage") at any given phase in the cycle. We can imagine that in addition to the varying "real" voltage, there is also a varying "imaginary voltage" such that there is a constant "complex voltage" having a radius of five. Of course, no one is ever going to power their refrigerator using imaginary voltages, but consider that the complex coordinates of any point on this circle will tell us both the voltage and its rate of change. When calculating the voltage in an AC circuit at any given moment, electrical engineers will of course throw out the imaginary portion of the complex number which describes the voltage at that moment. But if they are interested in the *phase* of the circuit, the imaginary portion carries useful information. They can use the complex number as a vector from the origin and calculate the angle this vector makes from the vertical axis. This angle is the phase of the voltage.

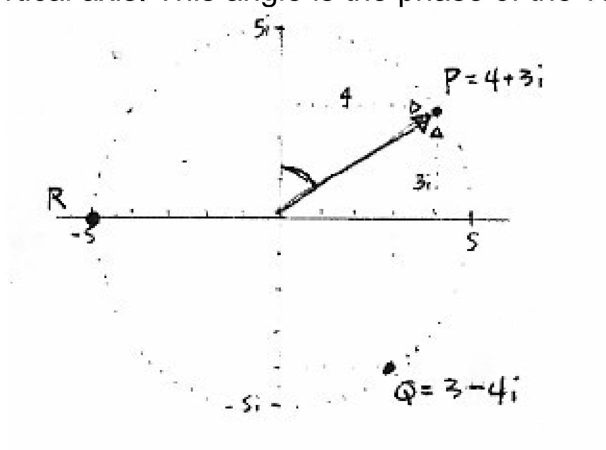

Figure 9-19.

If we assume that this graph is circular, we can also calculate the AC cycle's peak voltages using any single coordinate pair. Let's pick the coordinate pair (3, -4i). The vectors 3 and -4i form two sides of a right triangle, the other side of which has the length and direction which is the vector from the origin to the point (3,-4i). We can calculate this length. The Pythagorean theorem tells us that the square of the distance from the origin to this point is the sum of the squares of each coordinate. Three squared is nine. Minus four squared is sixteen. Nine and sixteen are twenty-five, the square roots of which are five and minus five. Five and minus five are our peak voltages. The distance from the origin to (3,-4i) is five. If we repeat this exercise for any coordinate on the circle, we will get the same distance calculation and the same two square roots which are our peak voltages.

Table 9-1.

	x	y	$\sqrt{(x^2+y^2)}$
P	4	3i	± 5
Q	3	-4i	± 5
R	-5	0	± 5

You might have noticed that I did not square the *i* when I calculated the distance from the origin to the point (3,-4*i*). If I had, my results would have been quite different. Since *i* squared is -1, -4*i* squared is -16. If I added that number to the square of 3 , the answer would have been the square root of minus seven rather than the square root of 25. And the answer would have varied from point to point rather than being constant like the radius of our circle.

To make the Pythagorean theorem work with our system of imaginary numbers, we need to make an adjustment. In two dimensions, the theorem takes the form $a^2 + b^2 = c^2$. We use this formula to calculate the distance from the origin to any point in the Cartesian plane, or to transform Cartesian coordinates of x and y to the polar coordinate r. We would say $x^2 + y^2 = r^2$. In three dimensions, another term is simply added: $x^2 + y^2 + z^2 = r^2$.

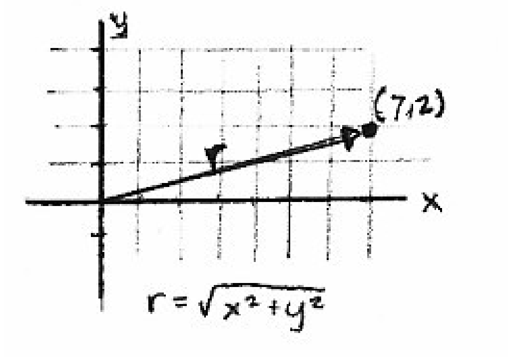

Figure 9-20.

This works for any number of dimensions. In the case where one of these dimensions is imaginary, though, let us agree that we change its sign from positive to negative. In other words, we subtract rather than add its squared value in our sum. Where b is the imaginary quantity, the general form of the equation would be:

$a^2 – ib^2 = c^2$

In the case of finding the radius r in a Cartesian coordinate system where x and y are real but z is imaginary, the calculation would be:

$x^2 + y^2 - iz^2 = r^2$

And as seen above, where voltage is a real vector and voltage change over time is an imaginary one, we used this formula:

$v^2 – i(v/t)^2 = r^2$

Only the calculation above will properly handle the imaginary term of our complex voltage coordinate. This convention of changing the sign of imaginary terms in Pythagorean sums will be important in our upcoming discussion of special relativity.

Table 9-2.

time	voltage
0	5i
1	3.5 + 3.5i
2	5
3	3.5 - 3.5i
4	-5i

5	-3.5 - 3.5i
6	-5
7	-3.5 + 3.5i
8	5i
9	3.5 + 3.5i
10	5

Note how the voltage cycles through the values of 5, -5i, -5, and 5i. We could just as easily have supposed our imaginary circuit to have lower peak voltages, so let's divide everything by five to see this cycle in its simplest terms: 1, -i, -1, and i. There is a symmetry between this cycle and what we saw earlier when we rotated the clock with the gum stuck to one hand: clockwise, right-to-left, counterclockwise, left-to-right. In each case, a full rotation took us through two interwoven sets of opposites. The series (1, -i, -1, i) is also repeated (although in reverse) in the exponential series of i:

$$i^0 = 1$$
$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$
. . .

The curve in Figure 9-16 is truly circular if and only if its radius is constant at all points. Recall that throughout this exercise, we have stipulated that the maximum slope of the voltage curve must have the same value as the peak voltage for this to be possible. When our peak voltage was five, we stipulated that the voltage must have a maximum slope of 5 volts per second. This requires that the alternation of voltage in the circuit to have a particular frequency. If we are above this frequency, the voltage will rise and fall at times more sharply than 5 volts per second; below this frequency, the voltage will never change at a rate as high as five volts per second. So our ability to calculate the peak voltage of the circuit from any complex coordinate pair rests on the assumption of a given frequency. Without offering proof, I submit to the reader that peak voltage may be inferred from any complex coordinate pair no matter what the frequency, as long as that frequency is known; and conversely that the frequency may be calculated from any coordinate pair as long as the peak voltage is known.

Let's look at some graphs of the same voltage with different frequencies. In Figure 9-17, the peak voltage is 5 and the frequency is 0.25 Hertz (cycles per second). The graph is elliptical because the voltage changes at a maximum of 10 volts per second. Figure 9-16 is a circular graph; here the frequency is lower, and the maximum rate of change in voltage is 5 volts per second. In Figure 9-18, the frequency is lower still, and the maximum rate of voltage change is 2.5 volts per second. Comparing the elliptical shapes in Figures 9-17 and 9-18 to the circular one in Figure 9-16, do you see how rotating the circle in Figure 9-16 on the vertical or horizontal axis would give it the elliptical shape seen in the other two? Appendix D contains some additional discussion of the mathematics of the ellipse and its relationship to other concepts already discussed.

In review: a single complex coordinate pair will describe a phase of the voltage and the instantaneous voltage level at that phase. If either the frequency or peak value of the voltage is also known, then the other value can be calculated. A circular or elliptical graph of complex coordinates can carry all of this information together: instantaneous voltage at each phase, peak voltage, and frequency.

But, you might say, doesn't a change in voltage create a varying magnetic field? Yes, it does, and the strength of that magnetic field is proportional to the rate of that change. What we have so far described as an “imaginary voltage” is the real magnetic field caused by the variance in the strength of the circuit's electrical field. The interplay between electricity and magnetism causes the electrons in an AC circuit to alternate between linear and angular movement. Electrical forces causes them to flow back and forth; magnetic forces cause them to spin clockwise and counterclockwise. The varying voltage is the rhythm of an electronic waltz: forward, left, backward, right . . . ; 1, -i, -1, i, . . .

Now we will open the black box which holds the power source for our circuit. Remember, there is a magnet rotating near a portion of the circuit. At one position, the magnet is causing current to flow in one direction; in the opposite position, current flows in the opposite direction. In between, the electrons spin one direction or the other. The magnet has a group of spinning electrons, just like the clock in chapter three had a spinning hand. The rotation of the magnet causes the electrons in the circuit to move sympathetically in the same ways as the gum on the clock when the clock is rotated: clockwise; right-to-left; counterclockwise; left-to-right.